

On the Log Quantile Difference of the Temporal Aggregation of a Stable Moving Average Process

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Abstract

A formula is derived for the log quantile difference of the temporal aggregation of some types of stable moving average processes, MA(q). The shape of the log quantile difference as a function of the aggregation level is examined and shown to be dependent on the parameters of the moving average process but not the quantile levels. The classes of invertible, stable MA(1) and MA(2) processes are examined in more detail.

Keywords: Quantile, Stable Distribution, Temporal Aggregation

1. Introduction

We recall some basic facts and definitions about stable moving average processes, temporal aggregation and log quantile differences.

1.1. Stable Moving Average Processes

Let $\{X_t\}$ be the moving average process of order q ,

$$X_t = \sum_{j=0}^q \theta_j e_{t-j} \quad (1)$$

where $\theta_0 = 1$ and $\{e_t\}$ is an independently and identically distributed (iid) sequence of stable random variables such that

$$e_t \sim S_\alpha^0(\beta^{(0)}, \gamma^{(0)}, \delta^{(0)}) \quad (2)$$

using the S^0 parameterisation of stable distributions in Nolan (1998). Let θ denote the $q + 1$ dimensional vector of moving average parameters

$$\theta = (\theta_0, \dots, \theta_q)' . \quad (3)$$

In (2) and the remainder of this paper, with the addition of various subscripts or superscripts, we use α, β, γ and δ to denote respectively the stability, skewness, scale and location parameters of a stable distribution. The S^0 parameterisation has the following useful properties.

(P1) If $Y \sim S_\alpha^0(\beta, \gamma, \delta)$, then for any $a \neq 0$,

$$aY + b \sim S_\alpha^0(\text{sign}(a)\beta, |a|\gamma, a\delta + b) \quad (4)$$

(P2) If Y_1, Y_2, \dots, Y_n are pairwise independent and $Y_j \sim S_\alpha^0(\beta_j, \gamma_j, \delta_j)$ for $j = 1, \dots, n$ then $\sum_{j=1}^n Y_j \sim S_\alpha^0(\beta, \gamma, \delta)$ where

$$\gamma^\alpha = \sum_{j=1}^n \gamma_j^\alpha, \quad \beta = \frac{\sum_{j=1}^n \beta_j \gamma_j^\alpha}{\sum_{j=1}^n \gamma_j^\alpha} \quad (5)$$

and

$$\delta = \begin{cases} \sum_{j=1}^n \delta_j + \tan(\pi\alpha/2) \left[\beta\gamma - \sum_{j=1}^n \beta_j \gamma_j \right] & \text{if } \alpha \neq 1 \\ \sum_{j=1}^n \delta_j + \frac{2}{\pi} \left[\beta\gamma \ln \gamma - \sum_{j=1}^n \beta_j \gamma_j \ln \gamma_j \right] & \text{if } \alpha = 1 \end{cases} \quad (6)$$

Properties (P1) and (P2) for $n = 2$ were given in Nolan (1998). The extension of Property (P2) to general n is a straightforward induction.

1.2. Temporal Aggregation

The temporal aggregation of the stochastic process $\{X_t\}$ is generally defined as the weighted sum of past and current process values. In this paper, we consider only a special case of temporal aggregation, sometimes referred to as flow aggregation, where all the weights equal 1. The flow aggregation of $\{X_t\}$ is given by

$$S_t^{(r)} = \sum_{i=0}^{r-1} X_{t-i}. \quad (7)$$

Henceforth, we refer to $\{S_t^{(r)}\}$ as the temporal aggregation of $\{X_t\}$ or the aggregated process, to r as the aggregation level and to $\{X_t\}$ as the base process.

We note that a moving average process is the temporal aggregation of an iid process and that the temporal aggregation of a moving average process is also a moving average process. A recent survey on temporal aggregation can be found in Silvestrini and Veredas (2008).

1.3. Log Quantile Difference

Let ξ_p denote the p th quantile of some distribution function. At quantile levels p_1, p_2 , such that $0 < p_1 < p_2 < 1$, we define the log quantile difference ζ_{p_1, p_2} to be

$$\zeta_{p_1, p_2} = \ln (\xi_{p_2} - \xi_{p_1}) . \quad (8)$$

We assume for the remainder of this paper, that any random variable on which a log quantile difference is calculated has a positive density at ξ_{p_1} and ξ_{p_2} . This assumption implies uniqueness of the quantiles ξ_{p_1} and ξ_{p_2} and that the log quantile difference is finite. Let us recall that the stable distributions satisfy this condition.

2. Log Quantile Difference of the Temporal Aggregation of a Stable Moving Average Process

Let $\{X_t\}$ be the moving average process of order q defined in (1) with stable innovations $\{e_t\}$, let $\{S_t^{(r)}\}$ denote the temporal aggregation of $\{X_t\}$ at aggregation level r and let $\zeta_{p_1, p_2}^{(r)}$ and $\zeta_{p_1, p_2}^{(0)}$ denote respectively the log quantile difference of $\{S_t^{(r)}\}$ and $\{e_t\}$ at quantile levels p_1, p_2 . In this section, we show under certain conditions on $\{X_t\}$, that

$$\zeta_{p_1, p_2}^{(r)} = \alpha^{-1} \ln \left(r \left| \sum_{i=0}^q \theta_i \right|^\alpha + g_\alpha (\theta) \right) + \zeta_{p_1, p_2}^{(0)} \quad (9)$$

where

$$g_\alpha (\theta) = \left(\sum_{i=0}^{q-1} \left| \sum_{j=0}^i \theta_j \right|^\alpha - q \left| \sum_{i=0}^q \theta_i \right|^\alpha + \sum_{i=1}^q \left| \sum_{j=i}^q \theta_j \right|^\alpha \right) \quad (10)$$

We start with a general result which applies to all stable moving average processes.

Theorem 1. *The distribution of the aggregated process $\{S_t^{(r)}\}$ is given by*

$$S_t^{(r)} \sim S_\alpha^0 (\beta^{(r)}, \gamma^{(r)}, \delta^{(r)}) \quad (11)$$

where

$$\gamma^{(r)} = \left(\sum_{j=0}^{r+q-1} |c_j|^\alpha \right)^{1/\alpha} \gamma^{(0)}, \quad \beta^{(r)} = \frac{\sum_{j=0}^{r+q-1} \text{sign}(c_j) |c_j|^\alpha}{\sum_{j=0}^{r+q-1} |c_j|^\alpha} \beta^{(0)}, \quad (12)$$

if $\alpha \neq 1$

$$\begin{aligned} \delta^{(r)} &= \left(\sum_{j=0}^{r+q-1} c_j \right) \delta^{(0)} + \\ &\quad \tan(\pi\alpha/2) \left[\beta^{(r)} \gamma^{(r)} - \beta^{(0)} \gamma^{(0)} \left(\sum_{i=0}^{r+q-1} \text{sign}(c_j) |c_j| \right) \right] \end{aligned} \quad (13)$$

if $\alpha = 1$

$$\begin{aligned} \delta^{(r)} &= \left(\sum_{j=0}^{r+q-1} c_j \right) \delta^{(0)} + \\ &\quad \frac{2}{\pi} \left[\beta^{(r)} \gamma^{(r)} \ln \gamma^{(r)} - \beta^{(0)} \gamma^{(0)} \left(\sum_{i=0}^{r+q-1} \text{sign}(c_j) |c_j| \ln(|c_j| \gamma^{(0)}) \right) \right] \end{aligned} \quad (14)$$

and

$$c_j = \begin{cases} \sum_{i=0}^j \theta_i & j = 0, \dots, q-1 \\ \sum_{i=0}^q \theta_i & j = q, \dots, r-1 \\ \sum_{i=j-r+1}^q \theta_i & j = r, \dots, r+q-1 \end{cases}. \quad (15)$$

Proof. From the definition of the aggregated process $\{S_t^{(r)}\}$, we have for $r \geq q$ that

$$\begin{aligned} S_t^{(r)} &= \sum_{i=0}^{r-1} X_{t-i} \\ &= \sum_{i=0}^{r-1} \sum_{j=0}^q \theta_j e_{t-i-j} \\ &= \sum_{j=0}^{r+q-1} c_j e_{t-j} \end{aligned} \quad (16)$$

where c_j is given by (15). An application of properties (P1) and (P2) proves the theorem. ■

Whilst Theorem 1 provides formulae for the stable distribution parameters of the aggregated process, in general it is not possible to derive from these a formula for the log quantile difference of the aggregated process. However, we can derive such a formula for those processes where $\beta^{(r)} = \beta^{(0)}$. To achieve this, we make use of the following lemma, which shows how the log quantile difference of a random variable is affected by linear transformations.

Lemma 2. *Suppose X is a random variable and $Y = aX + b$ for some $a > 0$ and $b \in \mathbb{R}$. Let $\xi_{X;p}$ and $\xi_{Y;p}$ denote respectively the p th quantile of X and Y , then*

$$\xi_{Y;p} = a\xi_{X;p} + b. \quad (17)$$

Let $\zeta_{X;p_1,p_2}$ and $\zeta_{Y;p_1,p_2}$ denote respectively the log quantile difference of X and Y at quantile levels p_1, p_2 , then

$$\zeta_{Y;p_1,p_2} = \ln a + \zeta_{X;p_1,p_2}. \quad (18)$$

Proof. By assumption we have

$$\begin{aligned} p &= P(X \leq \xi_{X;p}) \\ &= P(Y \leq a\xi_{X;p} + b) \end{aligned} \quad (19)$$

which proves (17) and (18) follows immediately. ■

We can now prove the formula for $\zeta_{p_1,p_2}^{(r)}$ in (9) under certain conditions on the base process, $\{X_t\}$.

Theorem 3. *If the base process $\{X_t\}$ satisfies either*

(A1)

$$\beta^{(0)} = 0 \quad (20)$$

or

(A2)

$$\sum_{j=0}^i \theta_j \geq 0 \quad \text{for } i = 0, \dots, q-1 \quad \text{and} \quad \sum_{j=i}^q \theta_j \geq 0 \quad \text{for } i = 1, \dots, q \quad (21)$$

then for $r \geq q$ the log quantile difference $\zeta_{p_1, p_2}^{(r)}$ is given by the formula in (9).

Proof. From Theorem 1, we have for $r \geq q$ that the aggregated process, $\{S_t^{(r)}\}$, has a stable distribution given by

$$S_t^{(r)} \sim S_\alpha^0(\beta^{(r)}, \gamma^{(r)}, \delta^{(r)}) \quad (22)$$

where $\beta^{(r)}, \gamma^{(r)}$ and $\delta^{(r)}$ are as shown in (12) and (13) or (14). If (A2) is satisfied, then all the c_j terms in (15) are non-negative and so

$$\text{sign}(c_j) |c_j|^\alpha = |c_j|^\alpha \quad \text{for } j = 0, \dots, r + q - 1. \quad (23)$$

Note that $\sum_{j=1}^q \theta_j \geq 0$ implies that $\sum_{j=0}^q \theta_j > 0$. Thus, if either (A1) or (A2) is satisfied, then

$$\beta^{(r)} = \beta^{(0)} \quad (24)$$

and $\{S_t^{(r)}\}$ is a scale and location transformation of the innovations $\{e_t\}$. Thus

$$\frac{S_t^{(r)} - \delta^{(r)}}{\gamma^{(r)}} \sim \frac{e_t - \delta^{(0)}}{\gamma^{(0)}} \quad (25)$$

and so from Lemma 2

$$\zeta_{p_1, p_2}^{(r)} = \ln(\gamma^{(r)}/\gamma^{(0)}) + \zeta_{p_1, p_2}^{(0)}. \quad (26)$$

Substituting (12) and (15) proves the theorem. ■

Although for our purposes the formula for $\zeta_{p_1, p_2}^{(r)}$ in (9) is only valid for integer values of $r \geq q$, nonetheless it is a function of r which is well-defined for all real positive values of r . Formally, we can take partial derivatives of $\zeta_{p_1, p_2}^{(r)}$ with respect to $\ln r$, to get for $r \geq q$

$$\frac{\partial}{\partial \ln r} \zeta_{p_1, p_2}^{(r)} = \alpha^{-1} \frac{r |\sum_{i=0}^q \theta_i|^\alpha}{r |\sum_{i=0}^q \theta_i|^\alpha + g_\alpha(\theta_1, \dots, \theta_q)} \quad (27)$$

and

$$\frac{\partial^2}{(\partial \ln r)^2} \zeta_{p_1, p_2}^{(r)} = \alpha^{-1} \frac{r |\sum_{i=0}^q \theta_i|^\alpha g_\alpha(\theta_1, \dots, \theta_q)}{(r |\sum_{i=0}^q \theta_i|^\alpha + g_\alpha(\theta_1, \dots, \theta_q))^2}. \quad (28)$$

and draw conclusions on the shape of $\zeta_{p_1, p_2}^{(r)}$.

Corollary 4. *Let $\{X_t\}$ be a $MA(q)$ process satisfying the conditions of Theorem 3. Then*

$$\lim_{r \rightarrow \infty} \frac{\partial}{\partial \ln r} \zeta_{p_1, p_2}^{(r)} = \alpha^{-1}. \quad (29)$$

For $r \geq q$,

$$\text{sign} \left(\frac{\partial^2}{(\partial \ln r)^2} \zeta_{p_1, p_2}^{(r)} \right) = \text{sign}(g_\alpha(\theta)) \quad (30)$$

and therefore

$$\begin{aligned} &\text{if } g_\alpha(\theta) > 0 \text{ then } \zeta_{p_1, p_2}^{(r)} \text{ is convex in } \ln r, \\ &\text{if } g_\alpha(\theta) = 0 \text{ then } \zeta_{p_1, p_2}^{(r)} \text{ is linear in } \ln r, \\ &\text{if } g_\alpha(\theta) < 0 \text{ then } \zeta_{p_1, p_2}^{(r)} \text{ is concave in } \ln r. \end{aligned} \quad (31)$$

Remark 5. *If $\beta^{(0)} \neq 0$ and any of the c_j terms in (15) are negative, then $\beta^{(r)} \neq \beta^{(0)}$. In that case, (25) and consequently (9) do not hold. In general, equality relations for the quantiles of the sums of random variables in terms of the quantiles of the summands are difficult to achieve. (Watson and Gordon (1986), Liu and David (1989))*

Remark 6. *In the special case where $\{X_t\}$ is iid, we have*

$$\begin{aligned} \gamma^{(r)} &= r^{1/\alpha} \gamma^{(0)}, \quad \beta^{(r)} = \beta^{(0)}, \\ \delta^{(r)} &= \begin{cases} r\delta^{(0)} + \tan(\pi\alpha/2) \beta^{(0)} \gamma^{(0)} (r^{1/\alpha} - r) & \text{if } \alpha \neq 1 \\ r\delta^{(0)} + \frac{2}{\pi} \beta^{(0)} \gamma^{(0)} r \ln r & \text{if } \alpha = 1 \end{cases} \end{aligned} \quad (32)$$

and the expression for $\zeta_{p_1, p_2}^{(r)}$ in (9) reduces to

$$\zeta_{p_1, p_2}^{(r)} = \alpha^{-1} \ln r + \zeta_{p_1, p_2}^{(0)}. \quad (33)$$

Note that the expressions for $\delta^{(r)}$ in (32) are different from those derived in Section 2.2 of Chan et al. (2008) which the author believes to be in error.

Remark 7. *The derivatives in (27) and (28) and therefore the results of Corollary 4 do not depend on p_1, p_2 for all $r \geq q$ and all α .*

In the next section we examine Corollary 4 in more detail for the special cases of invertible $MA(1)$ and $MA(2)$ processes.

3. Invertible Stable MA(1) and MA(2) Processes

An invertible MA(q) process is one where all roots of the polynomial

$$1 + \theta_1 z + \cdots + \theta_q z^q = 0 \quad (34)$$

lie outside the complex unit circle, $|z| > 1$. The region of \mathbb{R}^q in which invertible parameters reside is referred to as the invertibility region. The invertibility region of MA(1) processes is the set

$$\{\theta_1 : |\theta_1| < 1\}. \quad (35)$$

The invertibility region of MA(2) processes is the set

$$\{(\theta_1, \theta_2) : \theta_2 < 1 \text{ and } \theta_1 + \theta_2 > -1 \text{ and } \theta_1 - \theta_2 < 1\}. \quad (36)$$

Expressions for the invertibility region of higher order MA processes can be

found in Wise (1956). In this section we identify regions of the invertibility region of MA(1) and MA(2) processes where $g_\alpha(\theta)$ is either positive, zero or negative for various values of α . To conduct this analysis we require the following lemma.

Lemma 8.

$$\begin{aligned} \text{if } x, y > 0 \text{ and } 0 < \alpha < 1 & \text{ then } |x + y|^\alpha < |x|^\alpha + |y|^\alpha \\ \text{if } x, y > 0 \text{ and } \alpha = 1 & \text{ then } |x + y|^\alpha = |x|^\alpha + |y|^\alpha \\ \text{if } x, y > 0 \text{ and } 1 < \alpha \leq 2 & \text{ then } |x + y|^\alpha > |x|^\alpha + |y|^\alpha \end{aligned} \quad (37)$$

Proof. If a function f is strictly convex on (a, b) , then from Jensen's inequality for $a < x, y < b$

$$f(x) + f(y) < f(x + y) \quad (38)$$

The relations in (37) are proved by applying (38) to the function $f(x) = -(x^\alpha)$ for $0 < \alpha < 1$, and to the function $f(x) = x^\alpha$ for $1 < \alpha \leq 2$. ■

To assist this analysis we divide the invertibility region for an MA(2) process into 5 sub-regions as shown in Figure 1. These sub-regions are defined as open sets, so that the entire invertibility region consists of the union of the 5 sub-regions, the borders between them and the origin. The inequalities defining these sub-regions are listed in (39).

$$\begin{aligned}
\text{Sub-region 1} &= \{\theta : \theta_1 < -1 \text{ and } \theta_2 < 1 \text{ and } \theta_1 + \theta_2 > -1\} \\
\text{Sub-region 2} &= \{\theta : \theta_1 > -1 \text{ and } \theta_2 > 0 \text{ and } \theta_1 + \theta_2 < 0\} \\
\text{Sub-region 3} &= \{\theta : \theta_2 > 0 \text{ and } \theta_2 < 1 \text{ and } \theta_1 + \theta_2 > 0 \text{ and } \theta_1 - \theta_2 < 1\} \\
\text{Sub-region 4} &= \{\theta : \theta_2 < 0 \text{ and } -1 < \theta_1 + \theta_2 < 0 \text{ and } \theta_1 - \theta_2 < 1\} \\
\text{Sub-region 5} &= \{\theta : \theta_2 < 0 \text{ and } \theta_1 + \theta_2 > 0 \text{ and } \theta_1 - \theta_2 < 1\}
\end{aligned} \tag{39}$$

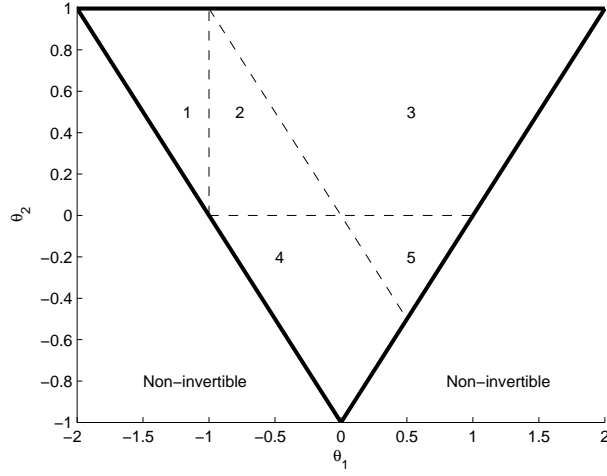


Figure 1: The 5 sub-regions of the invertibility region of an MA(2) process.

Remark 9. For an invertible MA(2) process, the set of values of (θ_1, θ_2) which satisfy condition (A2) in Theorem 3 consists of sub-region 3 and its borders with sub-regions 2 and 5.

The following theorem provides some properties of $g_\alpha(\theta)$ for θ in sub-region 3.

Theorem 10. If θ is an element of sub-region 3, then the function $g_\alpha(\theta)$ satisfies the following relations

$$g_\alpha(\theta) \text{ is } \begin{cases} > 0 & \text{if } 0 < \alpha < 1 \\ = 0 & \text{if } \alpha = 1 \\ < 0 & \text{if } 1 < \alpha \leq 2 \end{cases} \tag{40}$$

Proof. By definition for $q = 2$, we have,

$$g_\alpha(\theta) = 1 + |1 + \theta_1|^\alpha - 2|1 + \theta_1 + \theta_2|^\alpha + |\theta_1 + \theta_2|^\alpha + |\theta_2|^\alpha. \quad (41)$$

All θ in sub-region 3 satisfy $\theta_1 + \theta_2 > 0$, so using Lemma 8 we get for $0 < \alpha < 1$

$$|1 + \theta_1 + \theta_2|^\alpha < 1 + |\theta_1 + \theta_2|^\alpha. \quad (42)$$

All θ in sub-regions 3, satisfy $\theta_1 > -1$ and $\theta_2 > 0$, so using Lemma 8 we get for $0 < \alpha < 1$

$$|1 + \theta_1 + \theta_2|^\alpha < |1 + \theta_1|^\alpha + |\theta_2|^\alpha. \quad (43)$$

Therefore, for all θ in sub-region 3 and for $0 < \alpha < 1$ we have that $g_\alpha(\theta)$ is the sum of two strictly positive terms and so is strictly positive.

Similarly, for all θ in sub-region 3 and for $\alpha = 1$ we have that $g_\alpha(\theta)$ is the sum of two zero terms and so is zero. Finally, for all θ in sub-region 3 and for $1 < \alpha \leq 2$ we have that $g_\alpha(\theta)$ is the sum of two strictly negative terms and so is strictly negative. ■

Theorems similar to Theorem 10 for the other sub-regions and the borders between the sub-regions can be proven using the same approach. To cover all the sub-regions and borders of the invertibility region of an MA(2) process requires several such theorems. These are straightforward and are omitted from this paper.

A sub-region is said to be positive, zero or negative for a given α if $g_\alpha(\theta)$ is respectively positive, zero or negative for all points in the sub-region. A sub-region is said to be mixed for a given α if there exist some points in the sub-region for which $g_\alpha(\theta)$ is positive and other points for which $g_\alpha(\theta)$ is negative. Similar descriptions are used to describe the borders between the sub-regions. In Tables 1 and 2 we present a categorisation of all the sub-regions and the borders between the sub-regions using these descriptions.

	$0 < \alpha < 1$	$\alpha = 1$	$1 < \alpha \leq 2$
Positive Sub-regions	All	1,2,4,5	1,4
Zero Sub-regions	None	3	None
Negative Sub-regions	None	None	3
Mixed Sub-regions	None	None	2,5

Table 1: Categorisation of the sub-regions of the invertibility region of an MA(2) process into positive, zero, negative and mixed sub-regions with respect of $g_\alpha(\theta)$.

	$0 < \alpha < 1$	$\alpha = 1$	$1 < \alpha \leq 2$
Positive Borders	All	$(1, 2), (2, 4), (4, 5)$	$(1, 2), (2, 4), (4, 5)$
Zero Borders	None	$(2, 3), (3, 5)$	None
Negative Borders	None	None	$(2, 3), (3, 5)$

Table 2: Categorisation of the borders between the sub-regions of the invertibility region of an MA(2) process into positive, zero and negative borders with respect of $g_\alpha(\theta)$. We use (a,b) to denote the border between sub-regions a and b.

The set of invertible MA(1) processes is equivalent to the borders sub-regions 2 and 4 and between sub-regions 3 and 5. For iid processes $g_\alpha(\theta) = 0$ for all α . It is perhaps helpful to see the results of Tables 1 and 2 in graphical form as provided in Figure 2.

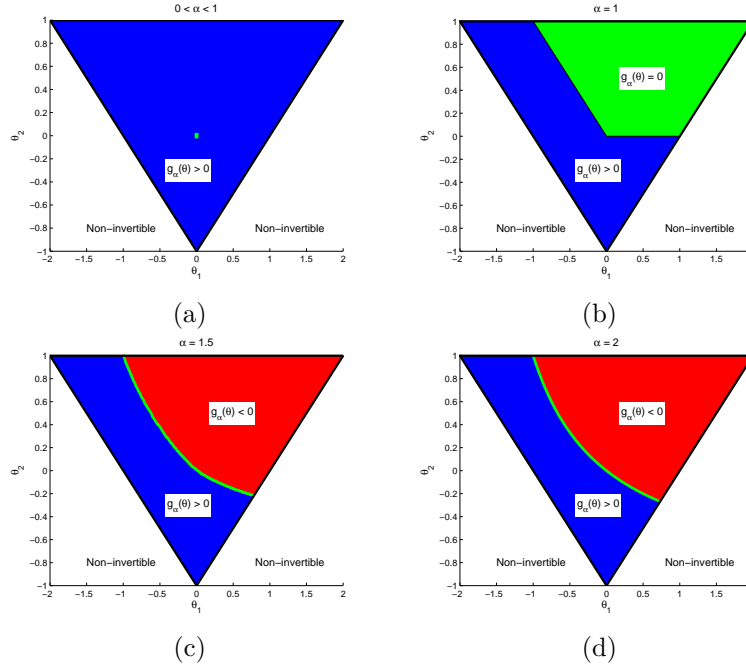


Figure 2: A graphical display of the categorisation of the invertibility region of MA(2) processes into positive (blue), zero (green) and negative (red) sub-regions for (a) $0 < \alpha < 1.0$, (b) $\alpha = 1.0$, (c) $\alpha = 1.5$ and (d) $\alpha = 2.0$.

Figure 2(a) is applicable to $g_\alpha(\theta)$ for any $\alpha \in (0, 1)$. Whilst Figures 2(c) and 2(d) appear similar, the locations of the respective green lines, i.e. the

sets

$$\mathcal{D}_\alpha = \{\theta : g_\alpha(\theta) = 0\}, \quad (44)$$

are not the same.

Remark 11. For an MA(2) process, it is straightforward to show that

$$\mathcal{D}_2 = \{\theta : \theta_1 + 2\theta_2 + \theta_1\theta_2 = 0\}. \quad (45)$$

For $1 < \alpha < 2$, closed form expressions for \mathcal{D}_α have not been obtained except to note that \mathcal{D}_α contains the points $\theta = (1, 0, 0)'$ and $\theta = (1, -1, 1)'$. Strictly $\theta = (1, -1, 1)'$ is on the border of, but not in the invertibility region.

To illustrate the behaviour of $\zeta_{p_1, p_2}^{(r)}$ where θ lie in different sub-regions of the invertibility region, we present plots of $\zeta_{0.50, 0.95}^{(r)}$ for various combinations of θ_1, θ_2 and α in Figure 3.

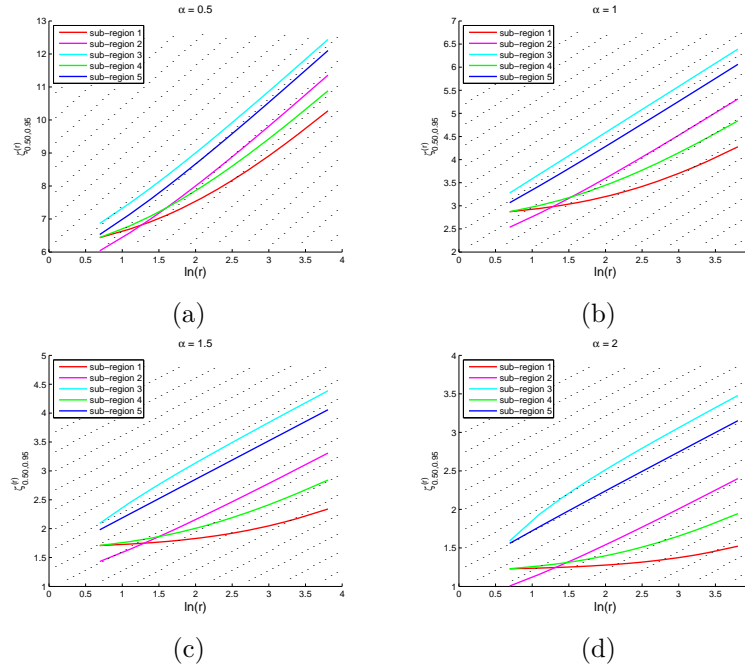


Figure 3: Plots of $\zeta_{0.50, 0.95}^{(r)}$ against the logarithm of the aggregation level for various symmetric stable MA(2) processes satisfying the conditions on Theorem 1.

For each sub-figure in Figure 3, we choose for sub-region 1: $(\theta_1, \theta_2) = (-1.4, 0.6)$, sub-region 2: $(\theta_1, \theta_2) = (-0.5, 0.2)$, sub-region 3: $(\theta_1, \theta_2) = (0.2, 0.9)$,

sub-region 4: $(\theta_1, \theta_2) = (-0.2, -0.4)$ and sub-region 5: $(\theta_1, \theta_2) = (0.7, -0.2)$. The dotted parallel lines in Figure 3 have a slope $1/\alpha$.

As shown in Corollary 4, for each choice of α, θ in Figure 3, it can be seen that the plot of $\zeta_{0.50, 0.95}^{(r)}$ against $\ln r$ is concave, linear or convex wherever $g_\alpha(\theta)$ is negative, zero or positive and that the sign of $g_\alpha(\theta)$ agrees with the results in Table 1. In all cases the derivative $\partial \zeta_{p_1, p_2}^{(r)} / \partial \ln r$ approaches $1/\alpha$ with increasing r . The convergence of the derivative $\partial \zeta_{p_1, p_2}^{(r)} / \partial \ln r$ to $1/\alpha$ can be much slower in the positive sub-regions than in the negative sub-regions. The example shown in Figure 3(d) for $\alpha = 2$ and sub-region 1, still has a derivative $\partial \zeta_{p_1, p_2}^{(r)} / \partial \ln r$ much less than $1/\alpha$ at an aggregation level of $\exp(3.8) \approx 45$.

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